Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

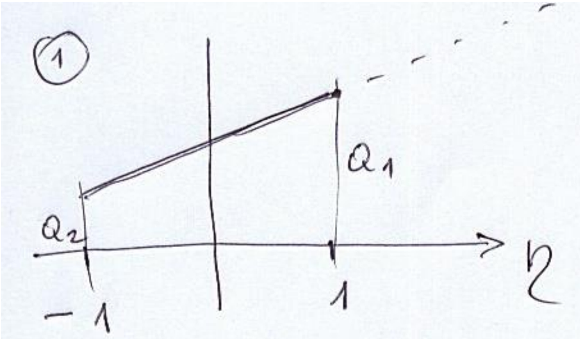
Finite element method (FEM1)

Lecture 5B. Numerical integration - examples

03.2025

Example 1. 1D case

Analytically:



$$f(\eta) = \frac{Q_1 - Q_2}{2} \cdot \eta + \frac{Q_1 + Q_2}{2}$$

$$\int_{-1}^{1} f(\eta)\, d\eta = \int_{-1}^{1} \left[\frac{Q_1 - Q_2}{2} \cdot \eta + \frac{Q_1 + Q_2}{2} \right] d\eta =$$

$$= \frac{1}{2} \cdot \frac{Q_1 - Q_2}{2} \cdot \eta^2 \Big|_{-1}^{1} + \frac{Q_1 + Q_2}{2} \eta \Big|_{-1}^{1} = 0 + \frac{Q_1 + Q_2}{2} (1 - (-1)) = Q_1 + Q_2$$

Numerical integration using 1 Gauss point:

$$\int_{-1}^{1} f(\eta)\, d\eta \qquad \underline{\underline{(\eta_1 = 0 \,;\, w_1 = 2)}}$$

$$= f(0) \cdot 2 = \frac{Q_1 + Q_2}{2} \cdot 2 = Q_1 + Q_2$$

Polynomial degree	Number of Gauss points	ξ_i	w_i
1	1	0	2
3	2	$-1/\sqrt{3}$ $+1/\sqrt{3}$	1 1
5	3	$-\sqrt{0.6}$ 0 $+\sqrt{0.6}$	5/9 8/9 5/9

Example 2. 1D case

Analytically : $f(\eta) = 2\eta^2 - 3$

$$\int_{-1}^{1} f(\eta)\,d\eta = \int_{-1}^{1} (2\eta^2 - 3)\,d\eta = \frac{2}{3}\eta^3\Big|_{-1}^{1} - 3\eta\Big|_{-1}^{1} =$$

$$= \frac{2}{3}\cdot 2 - 3\cdot 2 = \frac{4}{3} - 6 = -4\frac{2}{3}$$

Numerical integration using 2 Gauss points :

Polynomial degree	**Number of Gauss points**	ξ_i	w_i
1	1	0	2
3	2	$-{}^{1}/_{\sqrt{3}}$ $+{}^{1}/_{\sqrt{3}}$	1 1
5	3	$-\sqrt{0.6}$ 0 $+\sqrt{0.6}$	5/9 8/9 5/9

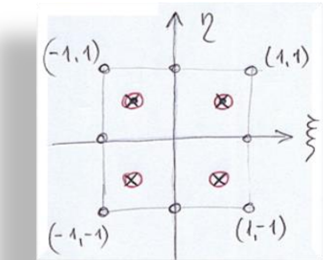
$$\int_{-1}^{1} f(\eta)\,d\eta = \sum_{i=1}^{2} w_i \cdot f(\eta_i)$$

$$= 1\cdot f\left(-\frac{1}{\sqrt{3}}\right) + 1\cdot f\left(\frac{1}{\sqrt{3}}\right) =$$

$$= 1\cdot\left(2\cdot\left(\frac{-1}{\sqrt{3}}\right)^2 - 3\right) + 1\cdot\left(2\cdot\left(\frac{1}{\sqrt{3}}\right)^2 - 3\right) = \frac{2}{3} - 3 + \frac{2}{3} - 3 =$$

$$= \frac{4}{3} - 6 = -4\frac{2}{3}$$

Example 3. 2D case



Analytically :

$$\int_{-1}^{1}\int_{-1}^{1}\left(\xi^3+\xi\eta+\eta^2\right)d\eta\, d\xi =$$

$$= \int_{-1}^{1}\left(\left(\xi^3\eta+\frac{1}{2}\xi\eta^2+\frac{1}{3}\eta^3\right)\Big|_{-1}^{1}\right)d\xi = \int_{-1}^{1}\left(2\xi^3+0+\frac{2}{3}\right)d\xi =$$

$$= \left(\frac{2}{4}\xi^4+\frac{2}{3}\xi\right)\Big|_{-1}^{1} = 0+\frac{2}{3}\cdot 2 = \boxed{\frac{4}{3}}$$

Polynomial degree	Num of GP	ξ_i	w_i
1	1	0	2
3	2	$-1/\sqrt{3}$ $+1/\sqrt{3}$	1 1

Numerical integration using 2x2 Gauss points :

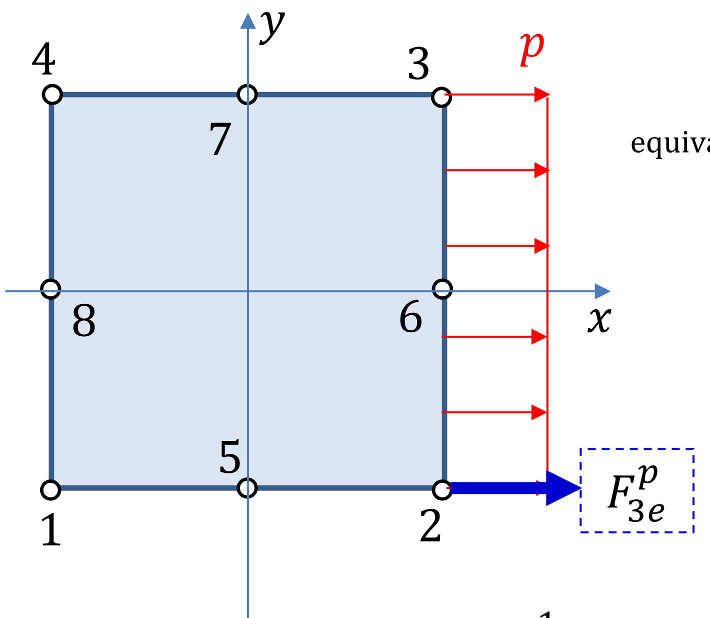
$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\, d\eta = \sum_{i=1}^{2}\sum_{j=1}^{2} w_i w_j f(\xi_i,\eta_j) \overset{\pm\frac{1}{\sqrt{3}}}{=}$$

$$= 1\cdot 1\cdot f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right) + 1\cdot 1\cdot f\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + 1\cdot 1\cdot f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right) + 1\cdot 1\cdot f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) =$$

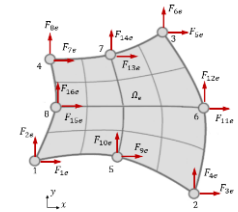
$$= \left[\left(-\frac{1}{\sqrt{3}}\right)^3+\frac{1}{3}+\frac{1}{3}\right] + \left[\left(-\frac{1}{\sqrt{3}}\right)^3-\frac{1}{3}+\frac{1}{3}\right] + \left[\left(\frac{1}{\sqrt{3}}\right)^3-\frac{1}{3}+\frac{1}{3}\right] + \left[\left(\frac{1}{\sqrt{3}}\right)^3+\frac{1}{3}+\frac{1}{3}\right] =$$

$$= \frac{2}{3}+\frac{2}{3} = \boxed{\frac{4}{3}}$$

Example 4. Determination of equivalent forces in an 8-node element (analytical vs numerical integration)



equivalent load vector due to surface load:



$$\lfloor F^{p} \rfloor_e = t_e \int_0^l \lfloor p \rfloor [N] \, ds$$

$$dy = \frac{l}{2} d\eta$$

$$F_{3e}^{p} = t_e \int_0^l p \, N_2 \, dy$$

$$N_2(1,\eta) = -\frac{1}{2}(1-\eta)\eta$$

Analytically: $F_{3e}^{p} = t_e p \int_{-1}^{1} [\frac{1}{2}(\eta-1)\eta] \; \frac{l}{2} \, d\eta = \frac{p\,l}{4} t_e \left(\frac{1}{3}\eta^3 - \frac{1}{2}\eta^2\right)\Big|_{-1}^{1} = \frac{p\,l}{6} t_e$

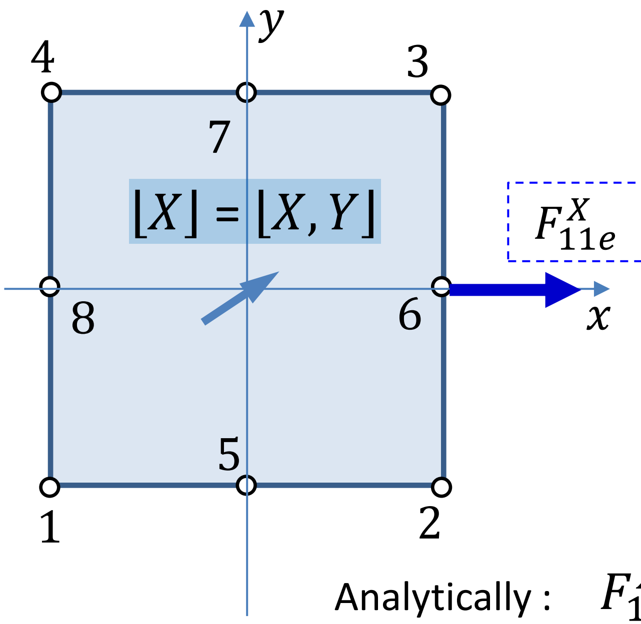
Numerical integration using 2 Gauss points:

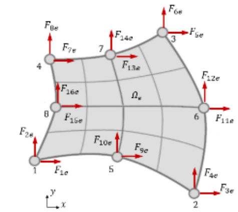
for 2 Gauss *points*: $\eta_1 = -\frac{1}{\sqrt{3}}$; $\eta_2 = \frac{1}{\sqrt{3}}$; $w_1 = w_2 = 1$

$$F_{3e}^{p} = \frac{1}{4} t_e p l \int_{-1}^{1} (\eta - 1)\eta \; d\eta = \frac{1}{4} t_e p l \left[1 \cdot \left(-\frac{1}{\sqrt{3}} - 1\right)\left(-\frac{1}{\sqrt{3}}\right) + 1 \cdot \left(\frac{1}{\sqrt{3}} - 1\right)\left(\frac{1}{\sqrt{3}}\right)\right] =$$

$$= \frac{1}{4} t_e p l \left[\frac{1}{3} + \frac{1}{\sqrt{3}} + \frac{1}{3} - \frac{1}{\sqrt{3}}\right] = \frac{p\,l}{6} t_e$$

Example 5. Determination of equivalent forces in an 8-node element (analytical vs numerical integration)





$dy = \frac{l}{2}\, d\eta$

$dx = \frac{l}{2}\, d\xi$

equivalent load vector due to mass load :

$$\lfloor F^X \rfloor_e = t_e \int_{A_e} \lfloor X \rfloor [N] dA_e$$

$$F_{11e}^{X} = t_e \int_{A_e} X\, N_6\, dA$$

$$N_6(\xi,\eta) = \tfrac{1}{2}(1+\xi)(1-\eta^2)$$

Analytically : $F_{11e}^{X} = t_e X \int_{-1}^{1}\int_{-1}^{1} \frac{1}{2}(1+\xi)(1-\eta^2)\, \frac{l}{2}\, d\xi\, \frac{l}{2}\, d\eta = \frac{1}{3} t_e X l^2$

Numerical integration using 2x2 Gauss points :

for 2 Gauss *points*: $\xi_1 = \eta_1 = -\dfrac{1}{\sqrt{3}}$; $\xi_2 = \eta_2 = \dfrac{1}{\sqrt{3}}$; $w_1 = w_2 = 1$

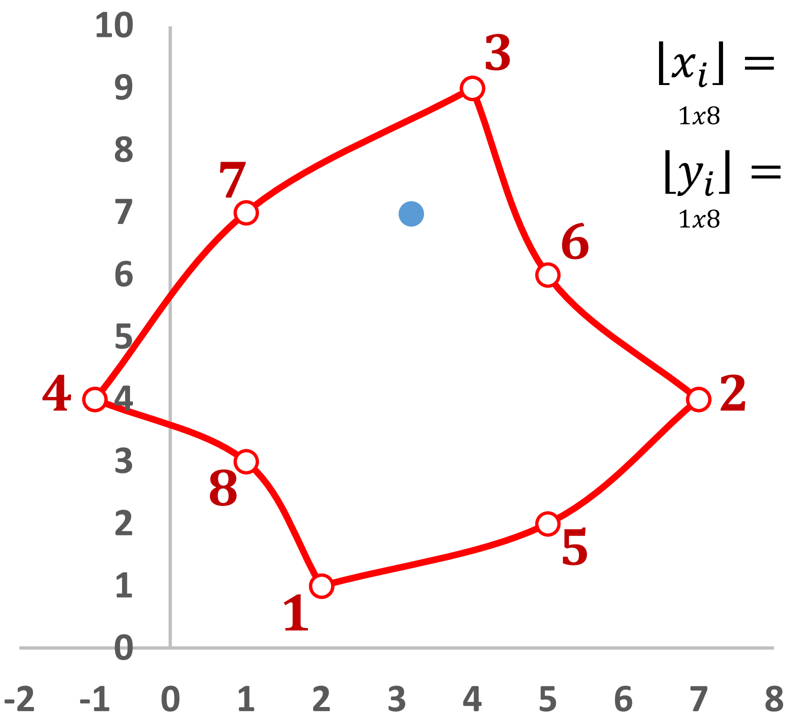
$$F_{11e}^{X} = t_e \frac{l^2}{8} X \int_{-1}^{1}\int_{-1}^{1} (1+\xi)(1-\eta^2)\, d\xi\, d\eta =$$

$$= t_e \frac{l^2}{8} X \left[1\cdot\left(1-\frac{1}{\sqrt{3}}\right)\left(1-\left(-\frac{1}{\sqrt{3}}\right)^2\right) + 1\cdot\left(1+\frac{1}{\sqrt{3}}\right)\left(1-\left(-\frac{1}{\sqrt{3}}\right)^2\right) + \right.$$

$$\left. +1\cdot\left(1-\frac{1}{\sqrt{3}}\right)\left(1-\left(\frac{1}{\sqrt{3}}\right)^2\right) + 1\cdot\left(1+\frac{1}{\sqrt{3}}\right)\left(1-\left(\frac{1}{\sqrt{3}}\right)^2\right)\right] = \frac{1}{3}\, t_e X l^2$$

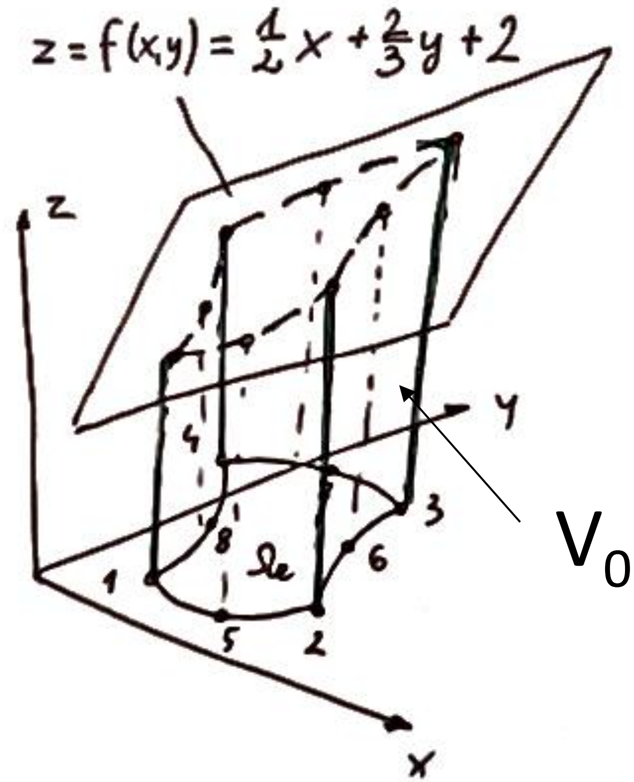
Example 6. 8-node element - numerical integration (n=3).

Find the area of the element and the volume V_0.



$$\underset{1x8}{\lfloor x_i \rfloor} = \lfloor 2, 7, 4, -1, 5, 5, 1, 1 \rfloor;$$

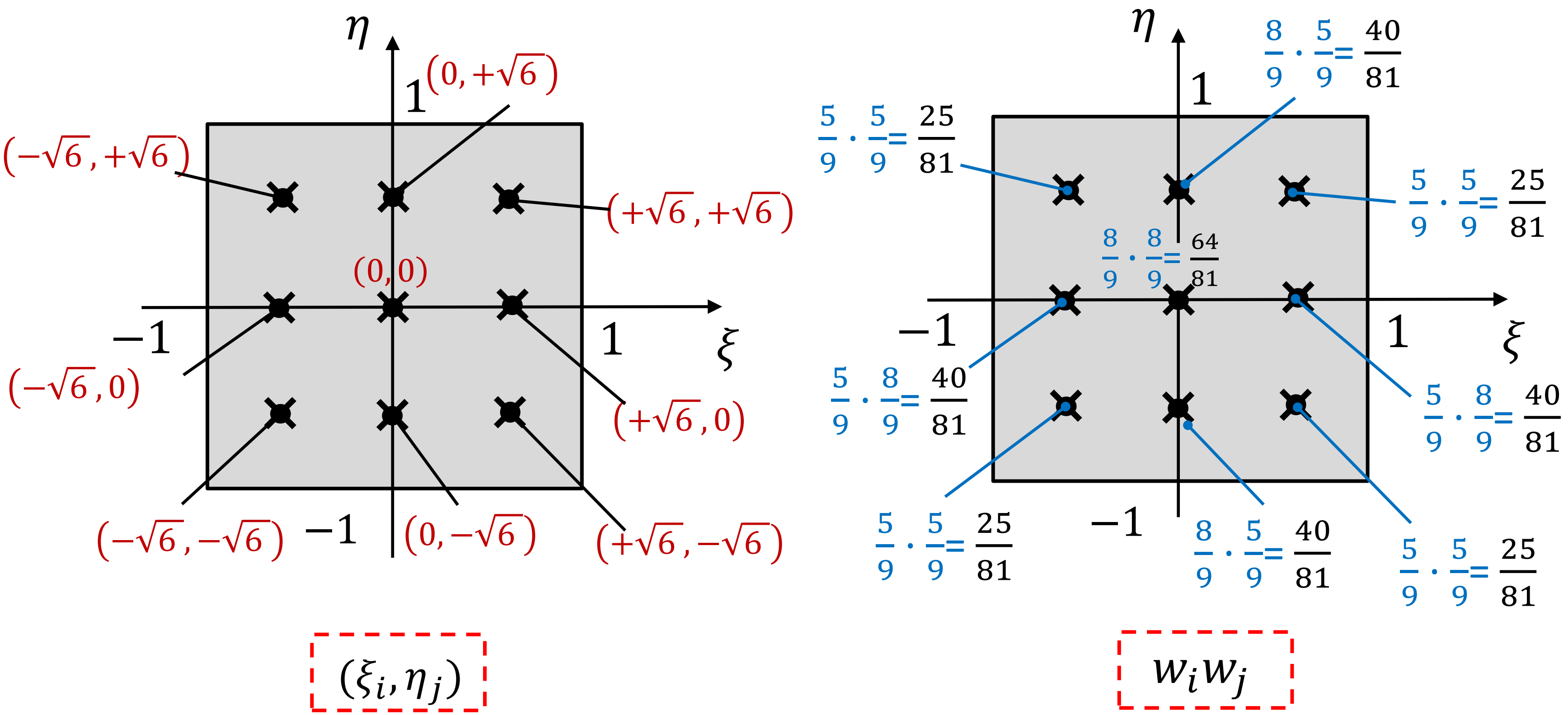
$$\underset{1x8}{\lfloor y_i \rfloor} = \lfloor 1, 4, 9, 4, 2, 6, 7, 3 \rfloor$$



Reminder: Gaussian quadrature rule for 2D elements

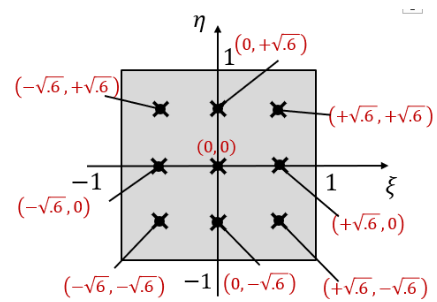
$$\boldsymbol{n = 3} \; : \; \xi_1 = \eta_1 = -\sqrt{0.6}\,, \quad \xi_2 = \eta_2 = 0 \quad , \quad \xi_3 = \eta_3 = \sqrt{0.6}$$

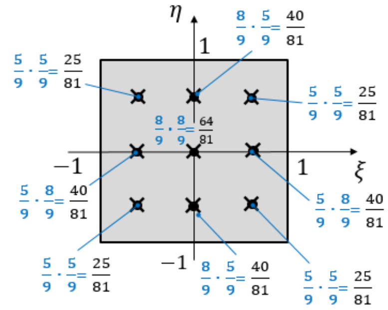
$$w_1 = w_3 = \frac{5}{9} \;\; ; \;\; w_2 = \frac{8}{9}$$



Volume calculation

$$x = \underset{1\times 8}{\lfloor N(\xi,\eta)\rfloor}\underset{8\times 1}{\{x_i\}_e} \quad ; \quad y = \underset{1\times 8}{\lfloor N(\xi,\eta)\rfloor}\underset{8\times 1}{\{y_i\}_e}$$





$$V_0 = \iint_A f(x,y)\,dx\,dy =$$

$$= \int_{-1}^{1}\int_{-1}^{1}\left(\frac{1}{2}x + \frac{2}{3}y + 2\right)\det[J(\xi,\eta)]\,d\xi d\eta =$$

$$= \int_{-1}^{1}\int_{-1}^{1}\left(\frac{1}{2}\lfloor N(\xi,\eta)\rfloor\{x_i\} + \frac{2}{3}\lfloor N(\xi,\eta)\rfloor\{y_i\} + 2\right)\det[J(\xi,\eta)]\,d\xi d\eta =$$

$$= \left|det[J] = \left(\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}\{x_i\}_e\right)\left(\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}\{y_i\}_e\right) - \left(\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}\{y_i\}_e\right)\left(\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}\{x_i\}_e\right)\right| =$$

The calculation of this determinant was shown in the presentation: ***Lecture 04B***

$$= \left(\frac{1}{2}\lfloor N(-\sqrt{.6},-\sqrt{.6})\rfloor\{x_i\} + \frac{2}{3}\lfloor N(-\sqrt{.6},-\sqrt{.6})\rfloor\{y_i\} + 2\right)\det\left[J(-\sqrt{.6},-\sqrt{.6})\right]\cdot\frac{5}{9}\cdot\frac{5}{9} +$$

$$+ \left(\frac{1}{2}\lfloor N(0,-\sqrt{.6})\rfloor\{x_i\} + \frac{2}{3}\lfloor N(0,-\sqrt{.6})\rfloor\{y_i\} + 2\right)\det\left[J(0,-\sqrt{.6})\right]\cdot\frac{8}{9}\cdot\frac{5}{9} +$$

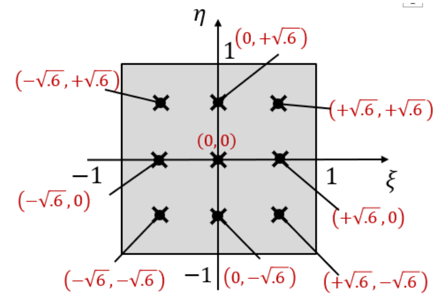
$$+ \left(\frac{1}{2}\lfloor N(\sqrt{.6},-\sqrt{.6})\rfloor\{x_i\} + \frac{2}{3}\lfloor N(\sqrt{.6},-\sqrt{.6})\rfloor\{y_i\} + 2\right)\det\left[J(\sqrt{.6},-\sqrt{.6})\right]\cdot\frac{5}{9}\cdot\frac{5}{9} +$$

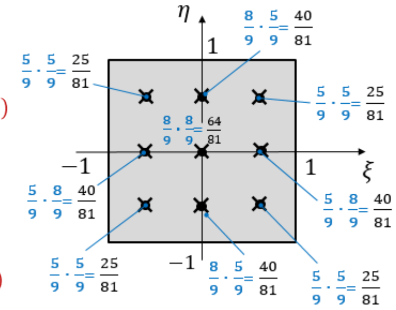
$$+ \left(\frac{1}{2}\lfloor N(-\sqrt{.6},0)\rfloor\{x_i\} + \frac{2}{3}\lfloor N(-\sqrt{.6},0)\rfloor\{y_i\} + 2\right)\det\left[J(-\sqrt{.6},0)\right]\cdot\frac{8}{9}\cdot\frac{5}{9} +$$

$$+ \left(\frac{1}{2}\lfloor N(0,0)\rfloor\{x_i\} + \frac{2}{3}\lfloor N(0,0)\rfloor\{y_i\} + 2\right)\det[J(0,0)]\cdot\frac{8}{9}\cdot\frac{8}{9} + \ \dots \ = 220.4\ mm^3$$

Area calculation

$$A = \iint_A dx\,dy =$$





$$= \int_{-1}^{1}\int_{-1}^{1} \det[J(\xi,\eta)]\, d\xi d\eta =$$

$$= \det\left[J(-\sqrt{.6}, -\sqrt{.6})\right]\cdot\frac{5}{9}\cdot\frac{5}{9} + \det\left[J(0, -\sqrt{.6})\right]\cdot\frac{8}{9}\cdot\frac{5}{9} + \det\left[J(\sqrt{.6}, -\sqrt{.6})\right]\cdot\frac{5}{9}\cdot\frac{5}{9} +$$

$$+\det\left[J(-\sqrt{.6}, 0)\right]\cdot\frac{8}{9}\cdot\frac{5}{9} + \det[J(0,0)]\cdot\frac{8}{9}\cdot\frac{8}{9} + \det\left[J(0, \sqrt{.6})\right]\cdot\frac{8}{9}\cdot\frac{5}{9} +$$

$$+\det\left[J(-\sqrt{.6}, \sqrt{.6})\right]\cdot\frac{5}{9}\cdot\frac{5}{9} + \det\left[J(0, \sqrt{.6})\right]\cdot\frac{8}{9}\cdot\frac{5}{9} + \det\left[J(\sqrt{.6}, \sqrt{.6})\right]\cdot\frac{5}{9}\cdot\frac{5}{9} = 33\,333\ mm^2$$